Spherical Paths

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Abstract:

Drawing on dozens of spherical objects and making some reflections on the geometry of the sphere, I discovered the existence of two infinite sets of figures that have been called "Spherical Paths" (or, more simply, SP): one set with the cardinality of the countable and the other set with the cardinality of the continuum. Then, studying their properties, I have adopted an analytical and a geometrical procedure for their construction. Furthermore, inside the second set, I have perceived an interesting subset of thirteen Platonic Spherical Paths, so called for being connected with the regular convex polyhedrons. For reasons which rely on the distinction between "real" and "fantastic", I will start this manuscript with a short story set in the bizarre world of Alice.

1. **A surprising event**

Recently Alice, inspired by the geometry of the sphere, spends her evenings drawing on the colored globes of her collection, often lingering on extravagant conjectures. Lately she ran into a problem that kept her awake all the night. Here are the facts of that evening:
Alice was tracing circles on a milky colored globe with the aim of testing an instrument of her own design, a special compass with intermediate nibs (figure 1), *suitable for operating on the surface of the sphere.*

Suddenly she had an idea, opened the compass with an amplitude L, pointed on the globe one of its extremities on A_1 *and, rotating it clockwise of an angle* ϑ*, she traced the arc* B_1B_2 (figure 2)*;clearly, each of the intermediate nips described an arch. Then, Alice pointed the compass in* B_2 *and , rotating it counterclockwise of a same angle, she traced the arc* A_1A_2 (figure 3). *This exercise was found so stimulating that she kept on describing other arcs following the same rules, alternating the two extreme pins of the compass as well as the clockwise and counterclockwise rotations, until this game came to an end: the compass, how strange!, appeared exactly in the initial position* (Figure 4).

With a candid smile Alice began to go around her globe to admire that beautiful set of arcs, She Decided to call it "Spherical Paths"[1] *(or, more simply, SP) and called "vertices" the alternative points of rotation of the compass.*

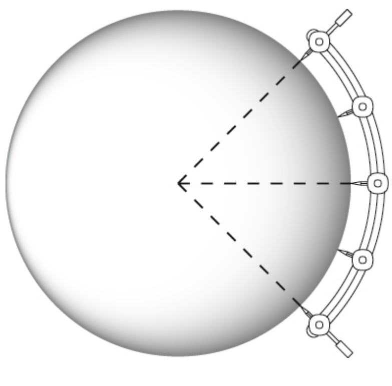

Fig. 1

[1] Note that in this paper the term "Path", when not otherwise specified, means a closed Path.

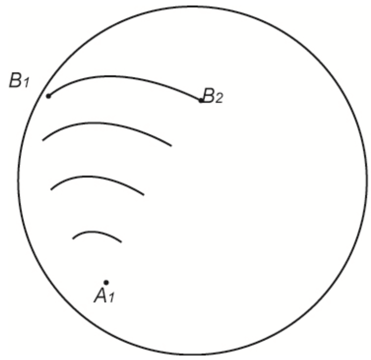

Fig. 2

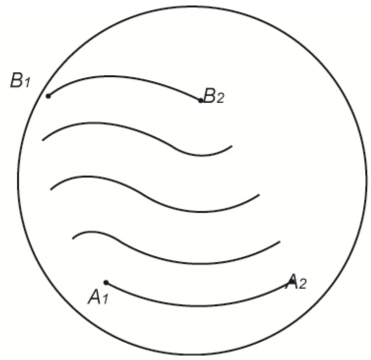

Fig. 3

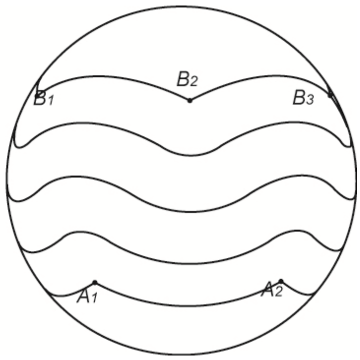

Fig. 4

In order to better visualize that SP characterized by ten vertices, she projected it into a plane[2] *(figure 5).*

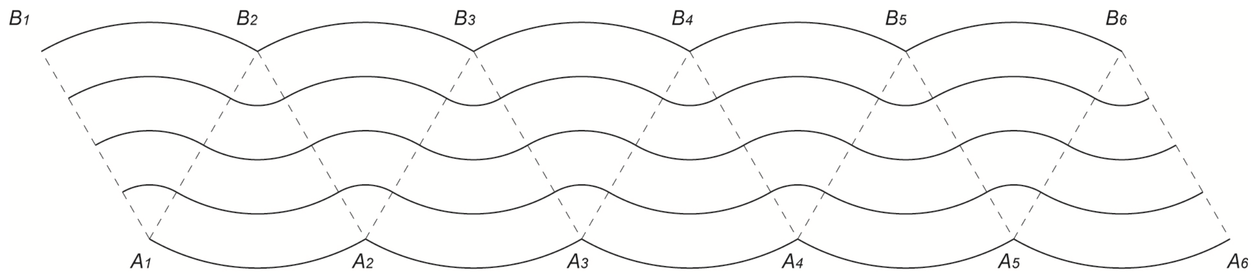

Fig. 5

That experience was very exciting and Alice tried to repeat it fixing a different value for L and ϑ, but this time the pins of the compass did not come back into the starting position (figure 6).

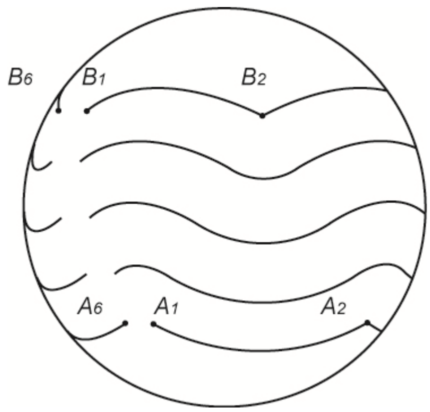

Fig. 6

[2] Even knowing that a figure on the sphere cannot be developed in the plane, Alice represented a conforming figure and made sure to respect the measures of some of its elements; in fact, the circular sectors of figure 5, all equal to each other, are consistent with those of the SP and their radius is of the same size as that on the sphere; moreover, the line segment between the extremes of the median arcs has the length of a maximum circle of the sphere and, on the latter, A_1 coincides with A_6 and B_1 with B_6. In the rest of the article, this type of representation will often be used.

Not being discouraged at all, she made many other attempts, each with new values of L and ϑ, but again without success. Anyway, she was somehow convinced that there should have been a procedure to obtain other Spherical closed Paths; this seemed to her an interesting problem to afford, but, shortly thereafter, it began to dawn and she fell asleep.

Three questions are here proposed to the reader:

1°- If on a sphere with radius chosen as unitary we try to build a SP in the same way followed by Alice, i.e. fixing arbitrarily the width of the compass as well as its rotation ϑ (for simplicity, in this first part of the story, we will always choose $L < \pi$ and $\vartheta \leq \pi$),[3] which will be the probability of success?

2°- Was the SP obtained by Alice in the beginning of the story a very exceptional fact?

3°- Which result would Alice have reached, after the failure of figure 6, in case she had continued to draw on the sphere a second series of arches equal to the first one?

To answer these questions (the solution is in Appendix at the end of the paper), the reader can find a help from the considerations made in the following sections 2 and 3.

2. **Analytical construction of an infinite set of SP with the cardinality of the countable.**

As well as Alice, we believe that the SP of figure 4 (resulting closed on the first try) is not the only possible one; therefore, we would like to find out a mathematical procedure that allows us to construct some others, choosing as unitary the radius of the sphere. For this purpose we will study that first SP[4] and its representation in the plane (figure 7).

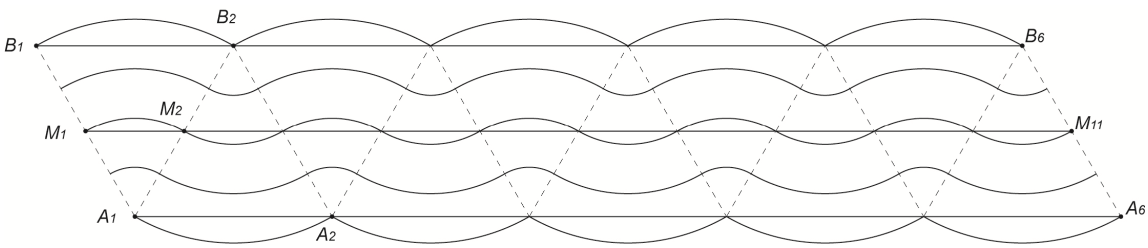

Fig. 7

In this figure we know that the straight line segment M_1M_{11} has the length of a maximum circle of the sphere[5] and that it is divided into ten parts of equal length p, which we call "step of the

[3] The choice of $L < \pi$ is justified by the fact that already for $L = \pi$ the compass would describe a point and thus there would be no Spherical Path. As for values of ϑ, on the other hand, Spherical Paths can also be obtained with values of ϑ taken in the interval $(\pi, 2\pi)$ but, in order not to burden the discussion, these will be taken into consideration only in the second part of the article.

[4] The radius of Alice's sphere is also assumed to be unitary.

[5] Since the straight line segments A_1A_6 and B_1B_6 represent two spherical circles equal to each other and equidistant from M_1M_{11}, the latter is the analogue of a great circle.

SP"; thus, it will be $10p = 2\pi$. A SP will be characterized, not only by the two real numbers *L and* ϑ, but also by two other numbers, one irrational, p, and one natural even,[6] n, such that

$$np = 2\pi, \quad (*)$$

from which it is $p = \frac{2\pi}{n}$. Let us examine how the relation (*) conditions the choices of L and ϑ studying the portion of the SP described by the first rotation of the compass,[7] that is the circular sector $B_1A_1B_2$ (figure 8), where $M_1M_2 = p$, $A_1B_1 = L$ e $B_1\hat{A}_1B_2 = \vartheta$. Tracing the bisector A_1C , we consider the right triangle M_1CA_1 (figura 9), where

$$M_1C = \frac{p}{2}, \; M_1A_1 = \frac{L}{2} \text{ e } M_1\hat{A}_1C = \frac{\vartheta}{2}.$$

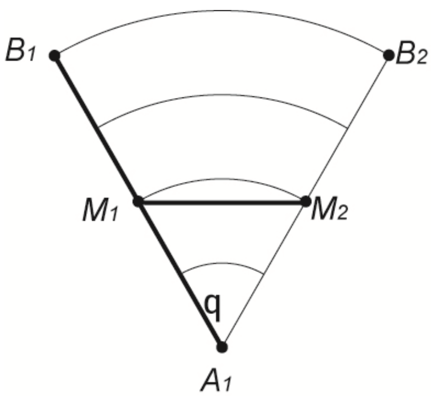

Fig. 8

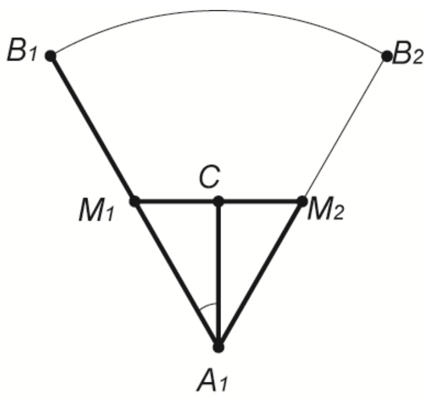

Fig. 9

From the trigonometry of the right triangle on the sphere it turns out[8] that $\sin\frac{p}{2} = \sin\frac{L}{2}\cdot\sin\frac{\vartheta}{2}$ and therefore, since $p = \frac{2\pi}{n}$, we will write:

$$\sin\frac{\pi}{n} = \sin\frac{L}{2}\cdot\sin\frac{\vartheta}{2} \qquad (**).$$

This implies that, fixing an n even with $n\geq 4$,[9] and choosing, for example, a value of L, the value of ϑ will result dependent from this latter and, precisely, it will be:

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[6] In the construction of any Spherical Path, a clockwise rotation of the compass will be followed by an counterclockwise one and, if you start for example with a clockwise rotation, at the end of the subsequent rotations you will have to end with a counterclockwise rotation, otherwise the compass, given the characteristics of the Spherical Path, it would not return to the starting point. The number of rotations must therefore be even and therefore also the number n of the steps associated with them.

[7] For simplicity we will refer in the figure to its analogue in the plan.

[8] In a right-angled spherical triangle, between sides a, b, c, relations similar to those concerning the sides of a plane right triangle hold. However, instead of the known relationships:

$$b = a \sin\vartheta, \; c = a \cos\vartheta$$

we will have, in a unitary sphere, the analogous:

$$\sin b = \sin a \sin\vartheta, \quad \sin c = \sin a \cos\vartheta.$$

For further clarifications see [1].

$$\vartheta = 2\arcsin\left(\frac{\sin\frac{\pi}{n}}{\sin\frac{L}{2}}\right)$$

and, from the condition that $-1 \leq \frac{\sin\frac{\pi}{n}}{\sin\frac{L}{2}} \leq 1$, it will be[10] $\sin\frac{\pi}{n} \leq \sin\frac{L}{2}$, from which $\frac{2\pi}{n} \leq L < \pi$. [11]

If, instead, we want to find L in function od ϑ, we will have:

$$L = 2\arcsin\left(\frac{\sin\frac{\pi}{n}}{\sin\frac{\vartheta}{2}}\right)$$

where we observe that it must be $\frac{2\pi}{n} < \vartheta \leq \pi$.

Summing up, the values of n, L and ϑ must satisfy the following conditions:

$$n \geq 4 \qquad \frac{2\pi}{n} \leq L < \pi \qquad \frac{2\pi}{n} < \vartheta \leq \pi.$$

If L and ϑ are fixed arbitrarily, we will be in the condition of affording the 1st question posed at the end of section 1. For example, fixing $L = \frac{\pi}{3}$ and $\vartheta = \frac{\pi}{2}$, we will have $\sin\frac{\pi}{n} = \sin\frac{\pi}{6} \cdot \sin\frac{\pi}{4}$, from which

$$\sin\frac{\pi}{n} = \frac{\sqrt{2}}{4}$$

$$n = \frac{\pi}{\arcsin\frac{\sqrt{2}}{4}} \cong 8{,}7;$$

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[9] In fact, if it were $n = 2$, the (**) would become $\sin\frac{\pi}{2} = \sin\frac{L}{2} \cdot \sin\frac{\vartheta}{2}$, from which $\sin\frac{L}{2} \cdot \sin\frac{\vartheta}{2} = 1$ and thus, $L = \vartheta = \pi$, while it has already been said that $L < \pi$ (see footnote 2)..

[10] Observe that in the Spherical Paths $\frac{\sin\frac{\pi}{n}}{\sin\frac{L}{2}} > 0$, since $\frac{\pi}{n}$, are between 0 and $\frac{\pi}{2}$.

[11] This stems from the fact that $\frac{\pi}{n}$, $\frac{L}{2} < \frac{\pi}{2}$; in fact, for a Spherical Path we have $n \geq 4$ e $L < \pi$;

thus, we cannot construct the corresponding SP, because the equation does not have a natural n as solution. In general, the expectation to obtain a SP starting from arbitrary choices of L and ϑ will depend on the probability to find an even solution of the equation (see answer in Appendix).

On the contrary, fixing n and L (or n and ϑ), it is always possible to obtain a corresponding SP.

Let us make an example: given $n = 8$ and $L = \dfrac{\pi}{3}$,[12] it will be

$$\vartheta = 2\arcsin\left(\frac{\sin\frac{\pi}{8}}{\sin\frac{\pi}{8}}\right) = 2\arcsin 1 = \pi .$$

Hereafter are shown the corresponding SP (figure 10) and its representation in the plane (figure 11).

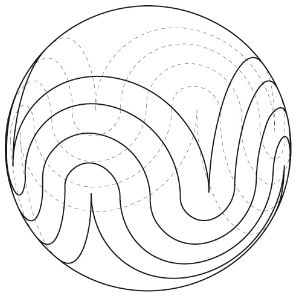

Fig. 10

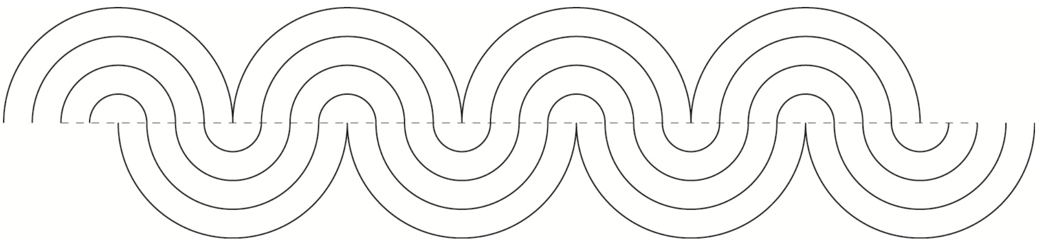

Fig. 11

In general, once fixed a value of n, it is possible to obtain a SP with $\vartheta = \pi$, for which it will be:

$$L = 2\arcsin\left(\frac{\sin\frac{\pi}{n}}{\sin\frac{\pi}{2}}\right) = 2\,\frac{\pi}{n}.$$

The Spherical Paths with $\vartheta = \pi$ will be called "Simple". These constitute an infinite set with the cardinality of the countable, they being are as many as the even natural numbers. The Simple SL are connectable to the regular polygons; in fact, linking the n consecutive vertices of the SP, we obtain a regular polygon with n sides; Therefore, from the SP of figure 10 we will obtain a regular octagon (figure 12) with its vertices on a maximum circle.

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[12] Observe that L satisfies the condition, since $\frac{2\pi}{n} \le L < \pi$..

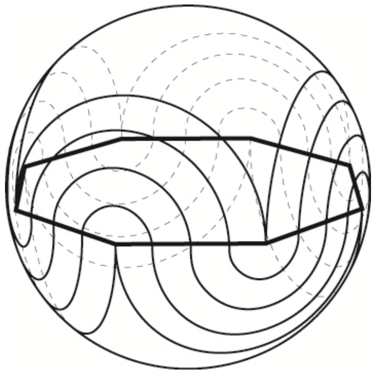

Fig. 12

Obviously, it is also possible to construct SP with $\vartheta<\pi$ (like the first one drawn by Alice); these latter will be called "Complex" and they constitute an infinite set too, but this, differently from the Simple set, has the cardinality of the continuum, since the Spherical Paths are as many as the values that L can assume in the interval $(\frac{2\pi}{n};\pi)$.[13] The Complex Spherical paths are connectable to the polyhedrons, since their n vertices are also the vertices of a convex polyhedron belonging to the family of the *antiprisms*.[14]

Let see an example of SP with n = 8 and $\vartheta = \frac{2\pi}{5}$, from which it is $L=2\arcsin\left(\frac{\sin\frac{\pi}{8}}{\sin\frac{\pi}{5}}\right)$ (figure 10.13); linking through straight line segments the vertices A_1 with A_2, A_2 with A_3..., A_4 with A_1, then B_1 with B_2, B_2 with B_3..., B_4 with B_1 and, finally, A_1 *with* B_1 and B_2, A_2 with B_2 and B_3…, A_4 with B_4 and B_1, we will obtain the corresponding polyhedron (figure 10.14) with 8 vertices, 10 faces and 16 edges.

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[13] Precisely they would be as many as they are contained in a countable infinity of intervals of this type without, however, changing the cardinality of the set which is always continuous.

[14] Given any complex Spherical Path of n vertices, the antiprism connected to it is a convex polyhedron with n vertices, $2n$ edges and $2 + n$ faces: two of these, called base surfaces of the antiprism, are regular polygons of $\frac{n}{2}$ equal sides between them, belonging to planes parallel to each other and rotated with respect to each other by an angle equal to $\frac{2\pi}{n}$, while the remaining n faces, which constitute the lateral surface, are triangles or equilateral (in which case the anti-prism is Archimedean) or isosceles, given that the antiprism can be dilated or contracted in the direction of the height as the wave amplitude varies (there is in fact isomorphism between this and the antiprism). Among the antiprisms there are two Platonic solids, the octahedron and the tetrahedron; however, the latter is an extreme case for which the aforementioned relationships between the number of vertices, edges and faces do not apply, as it does not have base surfaces (the tetrahedron has a pair of opposite edges as its basis. refer to [2].

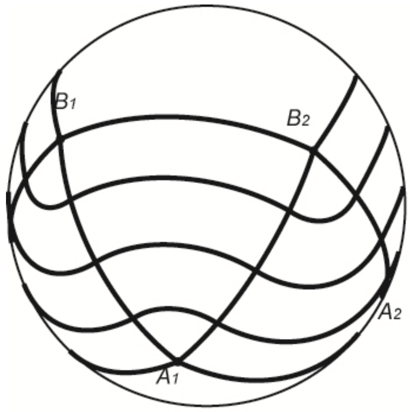

Fig. 13

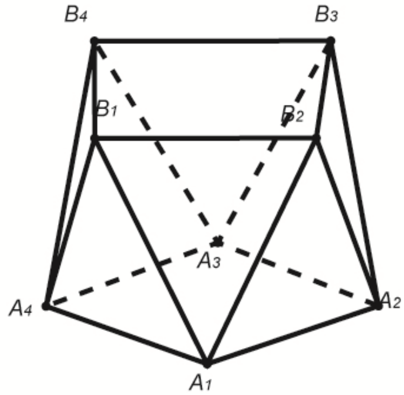

Fig. 14

3. **Geometrical construction of an infinite set of Spherical Paths with the cardinality of the continuum.**

If so far we have been concerned with finding the existence of Spherical Paths and the conditions under which they can be found, now we will try to provide an effective geometrical construction, once we have fixed the value of n; For this purpose, it should be sufficient, as done by Alice, to open the spherical compass to a width L (fixed in the interval $\frac{2\pi}{n} \leq L < \pi$), point it on the sphere and, rotating it by an angle $\vartheta = 2\arcsin\left(\frac{\sin\frac{\pi}{n}}{\sin\frac{L}{2}}\right)$ (individuated by a protractor capable of operating on our sphere!), trace the first curve and continue in analogous way adopted to construct figure 4, i.e. alternating the two extremities of the compass. Anyway, it is evident how hard such a construction would be if we consider the difficulty to open the compass by an angle located in the protractor that, in general, will be approximate. Therefore, we will propose a real procedure aimed to construct Spherical Paths with n steps. We are going to describe hereafter such a construction considering an example where $n = 8$:

1. *we divide the sphere into 8 equal spindles.*[15]

2. *we trace on the sphere the relative equator*

which will result divided into eight arcs.[16] Be $C_1, C_2, \ldots, C_8$ the middle points of these latter (figure 15).

[15] In this case it is possible to carry out this procedure with a spherical ruler and compass based on the Gauss theorem on the constructability of regular polygons with a number of sides $2k\, p_1, p_2, \ldots\, p_i$ with more Gauss primes [3], but in general this will not be possible for any even number. Therefore, when n does not allow the division of the sphere into equal spindles, we will have to be content with using the protractor.

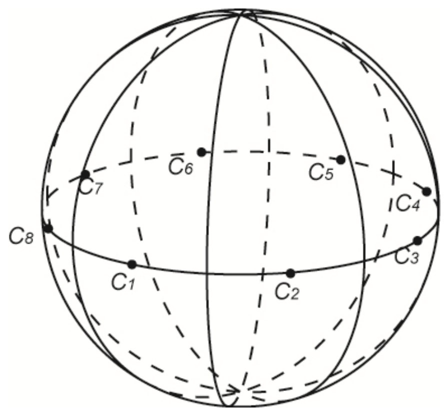

Fig. 15

3. *WE trace a geodesic arc that has its middle point in C_1 and its extreme points A_1 and B_1 on the two meridians of the spindle* (figure 16).

4. *Pointing the compass in A_1 and rotating it clockwise, we trace an arc from point B_1 to point B_2* (figure 17);

note that the geodesic arc A_1B_2 crosses the equator in the middle point C_2,[17] and thus it will represent the equivalent of A_1B_1 in the next spindle. The angles that A_1B_2 describe with the two meridians are congruent.[18]

5. *Then, pointing the compass in B_2 and rotating it counterclockwise from A_1 to the oint A_2 (on the next meridian between C_3 and C_4);*

an equal angle and, thus, a portion of the spherical surface equal to the first will be formed.

6. *We will keep proceeding in the same way;*

thus, 8 equal portions of the spherical surface corresponding to the expected SP will be formed (figure 18).

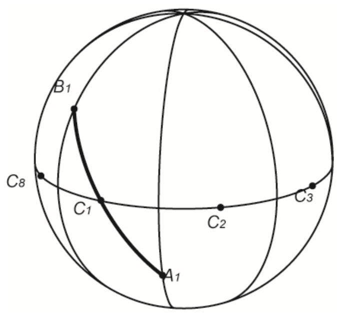

Fig. 16

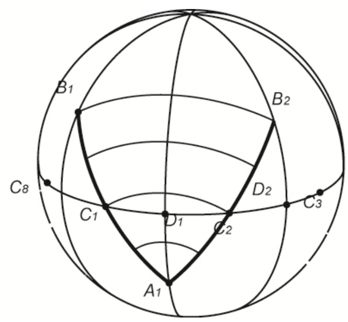

Fig. 17

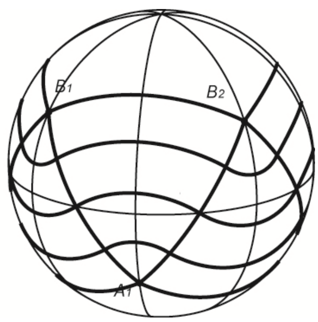

Fig. 18

[16] Each in length $p = \frac{2\pi}{8} = \frac{\pi}{4}$.

[17] In fact, in the triangle $C_1A_1C_2$ the meridian is the bisector of the angle at the vertex and, therefore, C_1, C_2 are symmetrical with respect to the meridian.

[18] Observe that the triangles $A_1D_1C_2$ and $B_2D_2C_2$ are congruent by the second congruence criterion (they are rectangles, $D_1C_2 = D_2C_2$ and, finally, the angles in C_2 are opposite).

This construction, beside being more practical than that one obtained by Alice, results "exact" when the spindle division of the sphere can be executed with spherical ruler and compass.[19] Moreover, it allows to clarify visually what has been already described analytically, i.e. the interdependency of the parameters L and ϑ, as well as the interval of the values that these can assume. In fact, referring to figure 15, if along the meridian where is A_1 we choose a point closer to the equator or further away, in the first case we will obtain $L < A_1B_1$ (as minimum $L = p = \frac{\pi}{4}$) and $\vartheta > B_1 \hat{A}_1 B_2$ (as maximum $\vartheta = \pi$) and, in the second case $L > A_1B_1$ (as upper extreme $L = \pi$) and $\vartheta < B_1 \hat{A}_1 B_2$ (as lower extreme $\vartheta = 0$).

Summing up, with $n = 8$ it is possible to construct Spherical Paths as you like choosing, with the criteria described above, either any L value such that $\frac{\pi}{4} \leq L < \pi$, or any ϑ value such that $\frac{\pi}{4} < \vartheta \leq \pi$ (and, in general, choosing either any L value such that $\frac{2\pi}{n} \leq L < \pi$, or any ϑ value such that $\frac{2\pi}{n} < \vartheta \leq \pi$).
Finally we have found the solution of the problem posed by Alice who, when she wakes up she will find her curiosity satisfied.

4. **Platonic Spherical Paths.**

The Spherical Paths described in this section will be called "**platonic**", because they are conceived starting from a radial projection of platonic solids on the sphere (the radius of which is chosen as unitary) constructed taking L equal to the measure of the spherical side of the solid we are considering and ϑ equal to the angle of its spherical face or to a n integer multiple of said angle(see footnote 2).

Here is a first example; given on the sphere the hexahedron (Figure 19), it will be possible to construct the corresponding SP having $L = A_1B_1$ and $\vartheta = B_1 \hat{A}_1 B_2$ (figure 20).

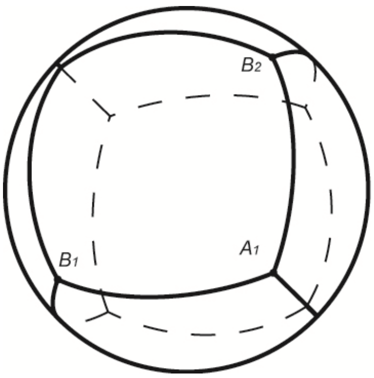

Fig. 19

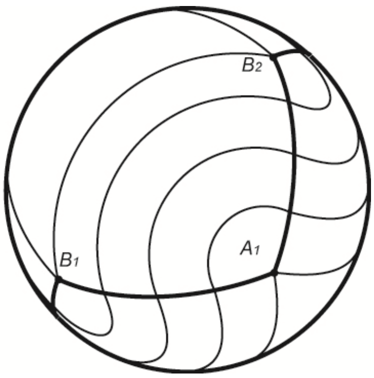

Fig. 20

[19] A spherical ruler, which is also useful for practical purposes, can be given by a hemispherical shell of a certain thickness whose concave surface has the same curvature of the sphere on which it will slide and be positioned. As for the compass, we have already seen it in figure 1.

The platonic Spherical Paths can be obtained if the values of L and ϑ found in relation with the platonic solids imply, in the equation

$$\sin\frac{\pi}{n} = \sin\frac{L}{2}\cdot\sin\frac{\vartheta}{2},$$

even values of n with $n \geq 4$.

To find the above values of L and ϑ we will introduce a preliminary study on the triangulation of the sphere that will allow us also a unitary visualization of the platonic solids and of the buildable related Spherical Paths. Such a study will be almost brief and, if the reader wants to avoid the deepening passages, he will not see compromised the understanding of the following study concerning the properties of the above Loops.

5. **Triangulation of the sphere.**

I will now describe two different triangulations, each obtained dividing the surface of the sphere of unit radius into an integer number of equal triangles[20] called "modules" The two above triangulations will be distinguishable from each other, because consisting of modules different in characteristics and number, and each coinciding with the ensamble of the radial projections of three among the five platonic solids[21] having their respective centers of gravity coinciding with the center of the sphere and their spherical faces (curvilinear regular polygons) formed by an integer number of modules.[22]

First Triangulation

We take the regular octahedron (figure 21) and project it radially onto the sphere that circumscribes it, so that it will be divided into eight identical curvilinear triangles (figure 22). In each of these we trace the curvilinear bisectors (figure 23) which, as already in the plane equilateral triangle, are medians and heights and, therefore, divide each triangle into six congruent triangles; the sphere will finally be subdivided into 48 curvilinear scalene triangles which we will indicate with M1.

[20] Said spherical triangles are congruent with each other and two by two symmetrical with respect to the plane passing through one of their sides in common and the center of the sphere (in the geometry of the sphere, as regards the curvilinear polygons, the same criteria of congruence of the plane geometry are valid).

[21] The projection of the octahedron will be common to both triangulations.

[22] It is our belief that there is no triangulation with equal triangles capable of providing the radial projection onto the sphere of all five Platonic solids. In any case, we would have resorted to the two triangulations proposed here, because they are useful for the construction of the Platonic Spherical Patgs and for the understanding of their mutual relations.

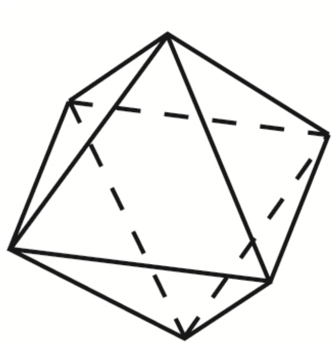

Fig. 21

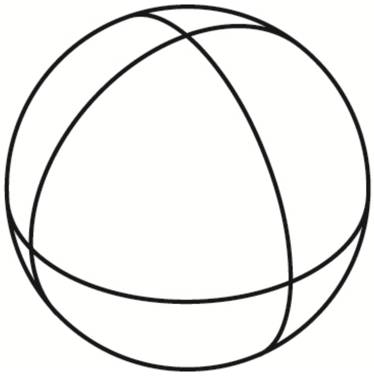

Fig. 22

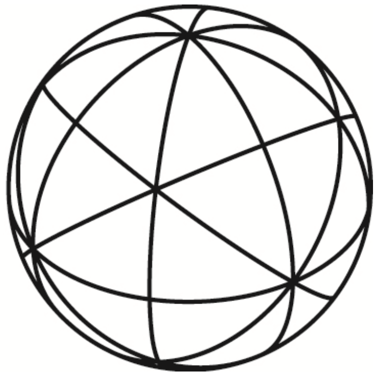

Fig. 23

This triangulation allows us to see on the same sphere the projections of the following regular convex polyhedra: of the aforementioned octahedron (where each of the 8 faces is formed, as we have seen, by 6 M1) (figure 24a), of the cube (6 curvilinear squares each formed by 8 M1) (figure 24b) and, finally, of the tetrahedron (4 curvilinear triangles each formed by 12 M1) (figure 24c).

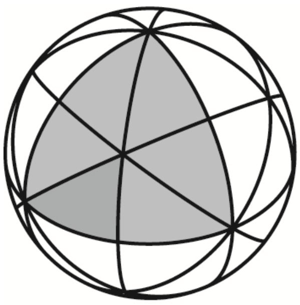

Fig. 24a

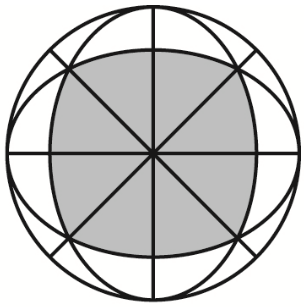

Fig. 24b

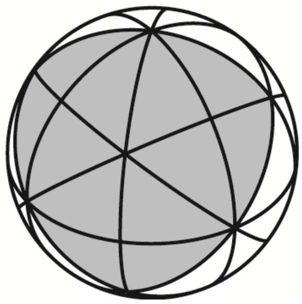

Fig. 24c

For further information we provide below the characteristics of module M1 (figure 25). Its angles are: $\alpha = \frac{\pi}{2}$, being the angle formed by the bisector which, in the equilateral triangle, is also height, $\gamma = \frac{\pi}{4}$, since it is the angle identified by the bisector, and finally $\beta = \frac{\pi}{3}$, since it is one of the six equal angles in which the rounded angle is divided by the bisectors at their meeting point.

Since the surface of the sphere has been divided into 48 equal parts, the S1 area of M_1 will be $S_1 = \frac{4\pi}{48} = \frac{\pi}{12}$, and this agrees with Gauss's formula,[23] in solid geometry known as the "Elegantissimum Theorema" [4].

23
$$\alpha + \beta + \gamma = \pi + \iint_T K\, dA$$

(where K is the Gaussian curvature $\frac{1}{r^2}$, which in a sphere of unit radius is $K = 1$) and which, therefore, in this case becomes: $\alpha + \beta + \gamma = \pi + S$ (where S is the triangle Area) and, since the sum of the angles of M_1 *is* $\alpha + \beta + \gamma = \frac{\pi}{2} + \frac{\pi}{3} + \frac{\pi}{4} = \frac{13\pi}{12}$, we have:

$$S = \frac{13\pi}{12} - \pi = \frac{\pi}{12}.$$

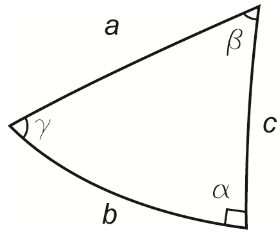

Fig. 25

As for the sides of M1, $b = \frac{\pi}{4}$ as half side of the three-right triangle which, in turn, is the fourth part of a great circle, from which $b = \frac{2\pi}{4} : 2 = \frac{\pi}{4}$. For the remaining sides it can be shown, with trigonometry, that $a = arcsin\sqrt{\frac{2}{3}}$ and that $c = \frac{\pi}{2} - arcsin\sqrt{\frac{2}{3}}$.

Let us now see the characteristics of the regular polygons projected onto the sphere based on the measurements of M1 and referring to figures24a, 24b, the curvilinear angles corresponding to the faces of the tetrahedron have side $L_T = 2\alpha = 2\,arcsin\sqrt{\frac{2}{3}}$ and angle $2\beta = \frac{2\pi}{3}$. The curvilinear squares corresponding to the faces of the hexahedron have side $L_E = 2c = 2(\frac{\pi}{2} - arcsin\sqrt{\frac{2}{3}})$ and angle $2\beta = \frac{2\pi}{3}$. Finally, the curvilinear triangles corresponding to the faces of the octahedron have side $L_O = 2b = \frac{\pi}{2}$ and angle $2\gamma = \frac{\pi}{2}$.

Second Triangulation

We take the regular icosahedron inscribed in the sphere (figure 26) and, as previously done with the octahedron, we project it radially onto it (figure 27) obtaining twenty identical equilateral curvilinear triangles with angles of $\frac{2}{5}\pi$, ince five converge at each vertex. In each of these we trace the bisectors (figure 28), so that on the sphere there will appear a total of 120 curvilinear scalene triangles which we will indicate with M2.

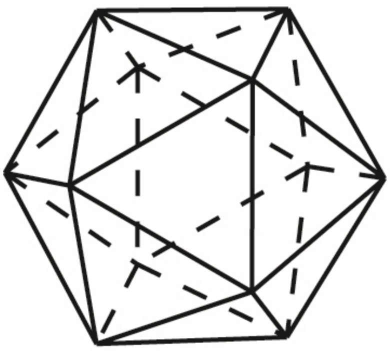
Fig. 26

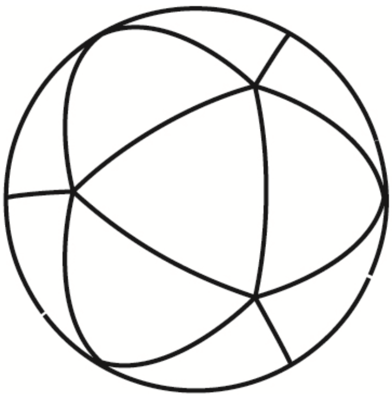
Fig. 27

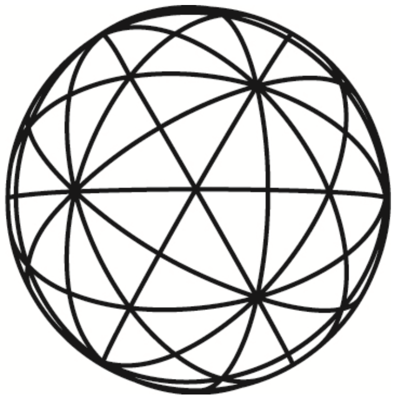
Fig. 28

With this triangulation we can identify the radial projection of the following regular polyhedra: of the aforementioned icosahedron (20 curvilinear triangles, each formed by 6 M2) (figure 29a), of the dodecahedron (12 curvilinear pentagons, each formed by 10 M2) (figure 29 b) and of the octahedron (8 curvilinear triangles, each formed by 15 M2) (figure 29c).[24]

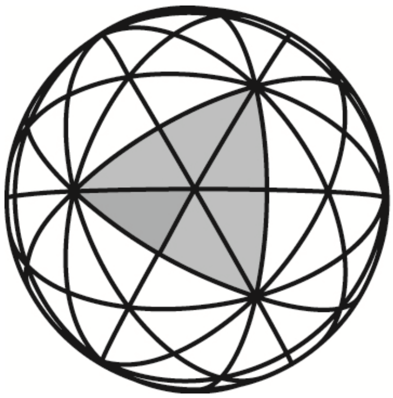

Fig. 29a

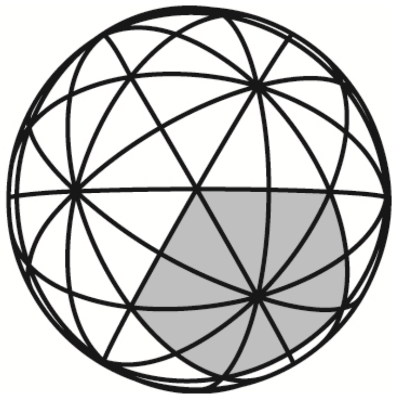

Fig. 29b

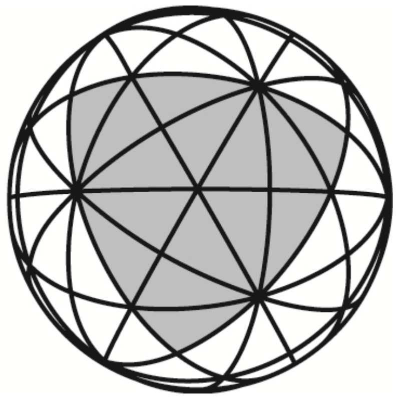

Fig. 29c

The angles of module M2 (figure 30) are: $\alpha = \frac{\pi}{2}$ and $\beta = \frac{\pi}{3}$ for the same reasons given for module M1, while $\delta = \frac{\pi}{5}$ because it is half the angle of the starting equilateral triangle. Since the spherical surface has been divided into 120 equal parts, the area S2 of the surface of M2 is S2 = $\frac{\pi}{30}$, according to Gauss's formula.[25]

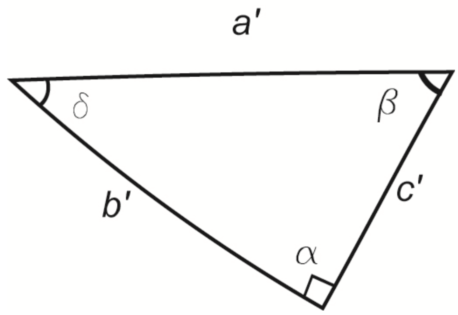

Fig. 30

As for the sides of M2, $a' = \frac{\pi}{2} - (b' + c')$; in fact, since each great circle is made up of $4a' + 4b' + 4c'$, it is $4(a'+b'+c') = 2\pi$ and, therefore, $a' + b' + c' = \frac{\pi}{2}$; it is then shown with trigonometry that $c' = \arcsin\frac{2}{\sqrt{3}+\sqrt{15}}$ and $b' = \arcsin\frac{2}{\sqrt{10+2\sqrt{5}}}$.[26]

---

[24] Also in this case (Fig. 29c) it is possible to see how the twelve-verticals of the icosahedron coincide with the centers of the faces of the dodecahedron and, viveversa, how the twenty vertices of the latter coincide with the centers of the faces of the icosahedron.

[25] Remembering that the surface area of module M_1 is $S_1 = \frac{\pi}{12}$, we shall have $S_1/S_2 = \frac{\frac{\pi}{12}}{\frac{\pi}{30}} = \frac{2}{5}$.

[26] From the measurements of the sides of M_1 and M_2 it results $\frac{a+b+c}{a'+b'+c'} = \frac{2}{3}$.

The curvilinear regular polygons present on the triangulated sphere with M2 have the following characteristics: the curvilinear triangles corresponding to the faces of the octahedron (see figure 29c) have side $L_O = \frac{\pi}{2}$, since $L_O = a' + b' + c'$, and inner corner $\alpha = \frac{\pi}{2}$; the curvilinear pentagons corresponding to the faces of the dodecahedron (see figure 29b) have side $L_D = 2c' = 2\,arcsin\frac{2}{\sqrt{3}+\sqrt{15}}$ and angle $2\beta = \frac{2\pi}{3}$. Finally, the curvilinear triangles corresponding to the faces of the icosahedron (figure 29a) have side $L_I = 2b' = 2\,arcsin\frac{2}{\sqrt{10+2\sqrt{5}}}$ and angle $2\delta = \frac{2\pi}{5}$.

6. **Constructability of the platonic Spherical Paths.**

Table 1 includes the characteristics of the faces of each platonic solid projected on the sphere of unitary radius.

Table 1

Platonic solid	***Spherical side***		**Angle of the spherical face**	
	Symbol	*Radians*	*Symbol*	*Radians*
Tetrahedron	L_T	$2\,arcsin\sqrt{\frac{2}{3}}$	ϑ_T	$\frac{2\pi}{3}$
Hexahedron	L_E	$2\,(\frac{\pi}{2} - arcsin\sqrt{\frac{2}{3}}\,)$	ϑ_H	$\frac{2\pi}{3}$
Octahedron	L_O	$\frac{\pi}{2}$	ϑ_O	$\frac{\pi}{2}$
Dodecahedron	L_D	$2\,arcsin\frac{2}{\sqrt{3}+\sqrt{15}}$	ϑ_D	$\frac{2\pi}{3}$
Icosahedron	L_I	$2\,arcsin\frac{2}{\sqrt{10+2\sqrt{5}}}$	ϑ_I	$\frac{2\pi}{5}$

The values of L and ϑ in this Table are such that the equation

$$\sin\frac{\pi}{n} = \sin\frac{L}{2}\cdot\sin\frac{\vartheta}{2}$$

that links each other is always satisfied by an even n (with $n \geq 4$) and precisely in the following order: $n_T = 4$, $n_H = 6$, $n_O = 6$, $n_D = 10$, $n_I = 10$.

In Table 2 here below, represented on the plane, are the Spherical Paths[27] buildable using the compass devised by Alice.

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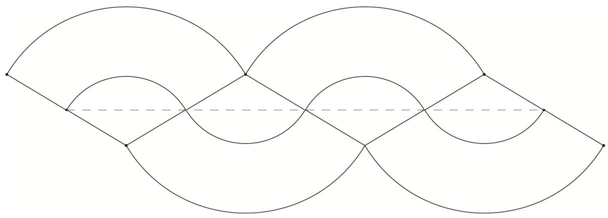
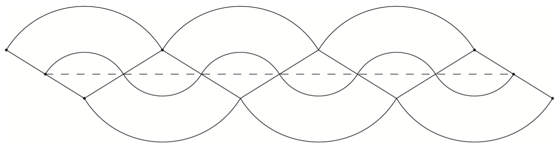
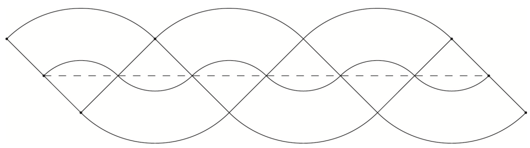
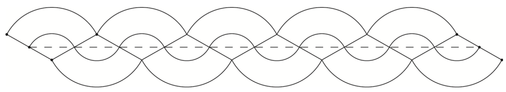
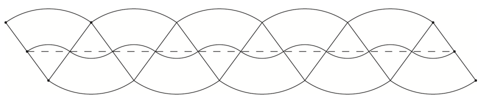
[27] For simplicity, the two intermediate curves will not be shown.

Table2

Platonic solid	**Spherical Path derived**[28]
Tetrahedron	
Hexahedron	
Octahedron	
Dodecahedron	
Icosahedron	

But there are a few others…

The five SWs described above are not the sole Platonic Paths (as initially defined). In fact, it is possible to construct a few others substituting the values of ϑ in Table 1 by integer multiples of the angle (see footnote 2) and leaving the values of L unchanged.

For example, we can obtain a sixth SP deriving it from the tetrahedron, choosing $L = L_T$ and $\vartheta = 2\,\vartheta_T = \frac{4}{3}\,\pi$. Proceeding in an analogous way with the other angles listed in Table 1, we will obtain other seven SP; thus, altogether there are thirteen platonic SL and their characteristics are summarized in the following Table.

[28]Since the dashed segment (which is divided by the central curve into n equal parts) is the analogue of a great circle on the sphere of unit radius, its length will be the same in all five of the Spherical Paths represented here.

Table 3

Spherical Path of	L	ϑ	n
Tetrahedron	L_T	$\vartheta_T = \frac{2}{3}\pi$	4
Tetrahedron	L_T	$2\vartheta_T = \frac{4}{3}\pi$	4
Hexahedron	L_H	$\vartheta_H = \frac{2}{3}\pi$	6
Hexahedron	L_H	$2\vartheta_H = \frac{4}{3}\pi$	6
Octahedron	L_O	$\vartheta_O = \frac{1}{2}\pi$	6
Octahedron	L_O	$2\vartheta_O = \pi$	4
Octahedron	L_O	$3\vartheta_O = \frac{3}{2}\pi$	6
Dodecahedron	L_D	$\vartheta_D = \frac{2}{3}\pi$	10
Dodecahedron	L_D	$2\vartheta_D = \frac{4}{3}\pi$	10
Icosahedron	L_I	$\vartheta_I = \frac{2}{5}\pi$	10
Icosahedron	L_I	$2\vartheta_I = \frac{4}{5}\pi$	6
Icosahedron	L_I	$3\vartheta_I = \frac{6}{5}\pi$	6
Icosahedron	L_I	$4\vartheta_I = \frac{8}{5}\pi$	10

The data of this Table allow us to focus some characteristic aspects of the platonic Spherical Paths. First of all, we note that in six of them we have $\vartheta < \pi$, in one $\vartheta = \pi$ and in the remaining six $\vartheta > \pi$. These latter, differently from the former which result already familiar to us, show intersections of their curves and, therefore, they are almost complicate. To have an idea of this, let us take a look of one of them, for example the SP of the tetrahedron with $\vartheta = \frac{4}{3}\pi$ (figure 31).[29]

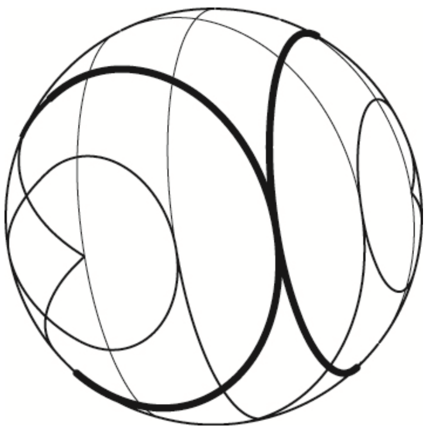


Fig. 31

[29] It is observed that in the two Spherical Paths of the tetrahedron the respective angles ϑ are supplementary, as well as in the following pairs of Spherical Paths: the two of the hexahedron, of the dodecahedron, the first and third of the octahedron, the first and fourth of the icosahedron and, finally, the second and third of the latter; this implies that each of said pairs of Spherical Paths connected to the same antiprism..

If we want to see it on the plane, as always done before, we realize that the above intersections, due to a loss of the isometry, can be represented partly and inadeguately (figure 32. Another characteristic of this particular Sç consists in covering the whole spherical surface.[30]

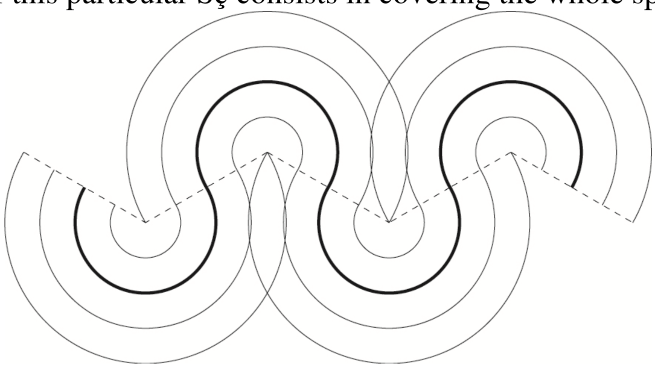


Fig. 32

We will now take a look at the sole platonic SP where $\vartheta = \pi$, that is the second of the octahedron in Table 3. This particular value of ϑ, as we already know, is telling us that it is a simple SP. It will be the last one to be illustrated (figure 33)[31] because endowed by a property that is not possessed by any other SP without intersections: the area of its surface coincides with that of the sphere.

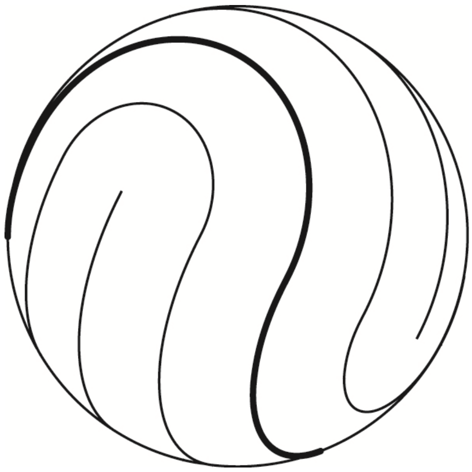


Fig. 33

Finally we will use its representation in the plane (figure 34) to make the following observations: since the amplitude L of the SP is equal to $\frac{\pi}{2}$, the four arcs B_1B_2, B_2B_3, A_1A_2, A_2A_3 will correspond to maximum semicircles of the sphere; but, knowing that B_1 coincides with B_3 and A_1 with A_3, the first of the above arcs will coincide with the third one, and the second with the

[30] Any SP, in which it is $\vartheta \geq \pi$ and $L \geq \pi$, will cover the entire surface of the sphere and, therefore, as can be deduced from Table 3, there will be only three Platonic Paths with such a characteristic, the one illustrated above, the second of the hexahedron and the second of the octahedron.

[31] Its central curve is highlighted in bold.

fourth one; thus, we will obtain only two maximum semicircles on the sphere belonging to two orthogonal planes opposite to the center of this latter (figure 33).

Furthermore, their extremes, which represent the 4 vertices of the Path, will also be the vertices of a square inscribed in a maximum circumference (which in Figure 34 corresponds to the dotted segment M_1M_5).

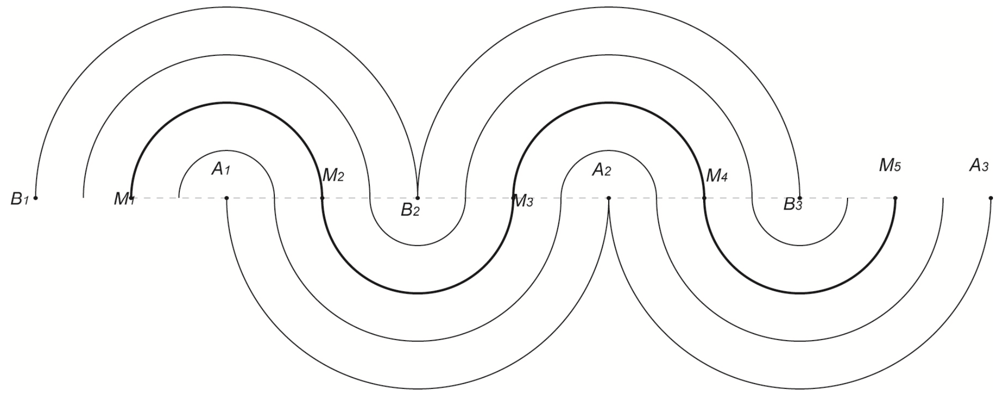


Fig. 34

Finally, from the data of Table 3 it is also inferred that only four Spherical Paths have the number of vertices equal to that of the solid from which they are respectively derived: the two of the tetrahedron and the two of the octahedron with $n = 6$; this implies that the vertices of the former will also be the vertices of a tetrahedron, while those of the latter will also be the vertices of an octahedron. So here is a question: to which solid will each of the other eight complex Platonic Paths be connected? The reader is left with the opportunity to find the answer and then compare it with the one reported in the appendix to the fourth point.

7.**The duality.**

Our research could thus be considered finished, given that we had essentially set ourselves two objectives: to find a way to construct the set of Spherical Paths and to classify their subsets. However, there is one last thing that would be worth knowing and that concerns the entire family of Platonic Spherical Paths. Since there is a precise relationship between these and the Platonic solids, one wonders whether the principle of duality which the latter obey can in any way be reflected on the former. In carrying out this brief investigation we will find it useful to refer to some figures relating to the previous study on the triangulation of the sphere.

Let's take the triangulated sphere with M1 of figure 24a and consider two sides of a spherical face of the octahedron, A_1B_1 and A_1B_2 (figures 35-36).

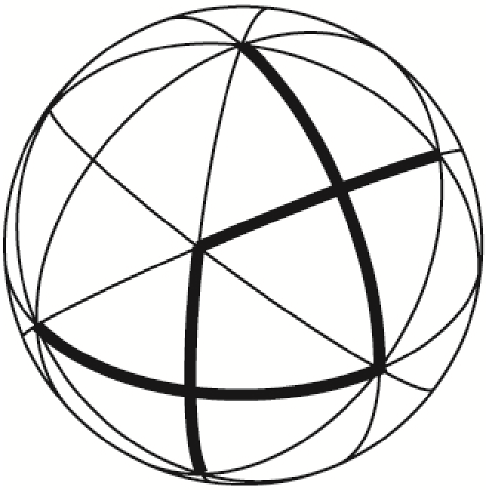


Fig. 35

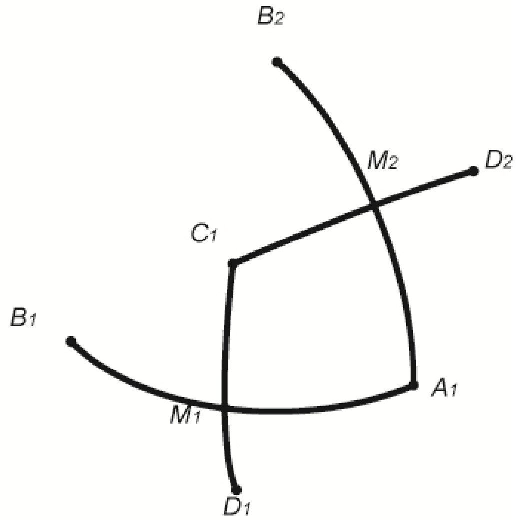


Fig. 36

Knowing that their midpoints, M_1 and M_2, of said sides coincide with the midpoints of the two consecutive sides C_1D_1 and C_1D_2 of a face of the cube, let's see what happens when we start building the SP of the octahedron: pointing the compass in $A1$ and tracing the arc B_1B_2 (figure 37), we observe that the arc M_1M_2 described by the central nib turns out to be ¼ of a circumference inscribed in the spherical face of the cube. You will have the reverse in building the SP of the hexahedron; in fact, pointing the compass at C_1 and tracing the arc D_1D_2 (Figure 38), the M_1M_2 arc described by the central nib will be different from the previous one and will be 1/3 of a circumference inscribed in the spherical face of the octahedron.

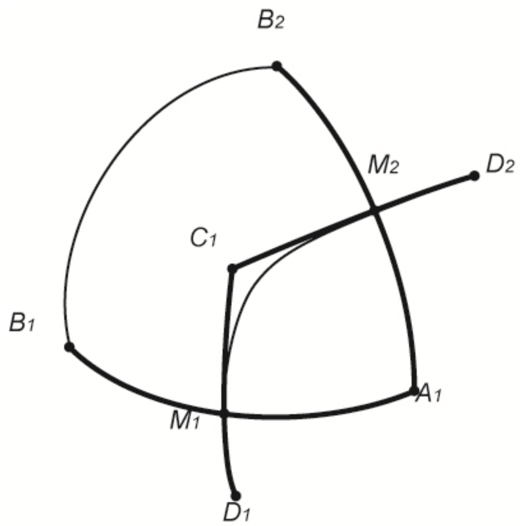


Fig. 37

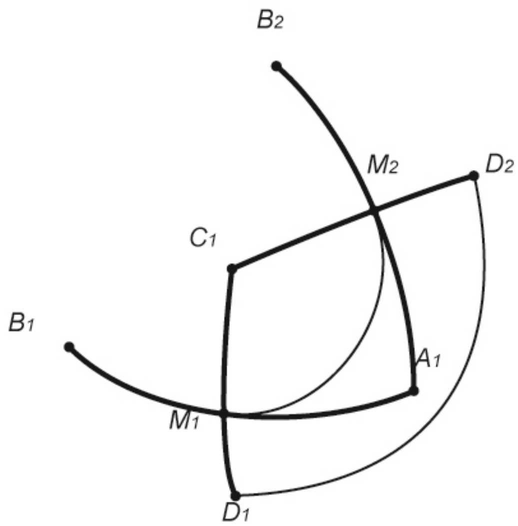


Fig. .38

This will apply to all bows described by the central nib. Therefore, the central curve of the SP of the octahedron (formed by six arcs of ¼ of a circumference and resulting inscribed in the faces of the hexahedron) we will call it "Loop Curve$_{1/4}$" (in short, "$LC_{1/4}$") of the hexahedron, while that of the SP of the hexahedron (formed by six arcs of circumference and resulting inscribed in six faces of the octahedron) we will call it "$LC_{1/3}$ of the octahedron". To get a clearer idea of the aforementioned central curves, we show them together with the solid in which they are respectively inscribed (Figures 39-40, where for brevity, CC stands for "Central Curve" and SL, as already established, stands for "Spherical Loop").

8. **The thirteen platonic Loop Curves (*LC*).**

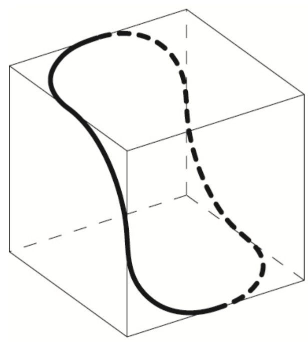


Fig. 39
$LC_{1/4}$ of the hexahedron
(*CC of the SP of the octahedron*)

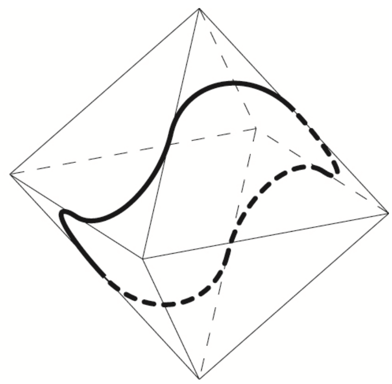


Fig. 40
$LC_{1/3}$ of the octahedron
CC of the SP of the hexahedron)

With the same argument, the central curve of the SL of the dodecahedron will be the $LC_{1/3}$ of the icosahedron (inscribed in ten of its faces, as shown in figure 41), while that of the SL of the icosahedron will be the $LC_{1/5}$ of the dodecahedron (inscribed on ten of its faces, as shown in figure 42). Otherwise, the central curve of the SP of the tetrahedron, which will be inscribed in the spherical faces of its dual tetrahedron, we will call $LC_{1/3}$ of the tetrahedron [6] (Figure 43).

All this is an implication of the principle of duality which the Platonic solids obey.

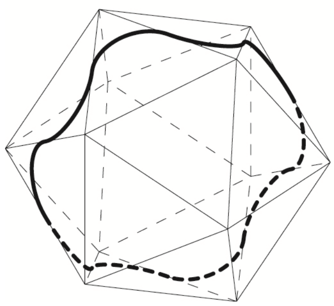


Fig. 41
$LC_{1/3}$ of the icosahedron
(*CC of the SP of the dodecahedron*)

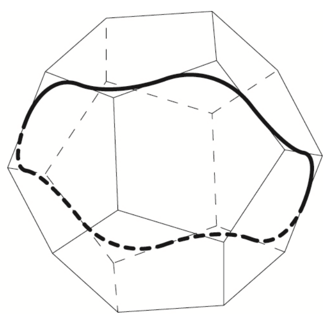


Fig. 42
$LC_{1/5}$ of the dodecahedron
(*CC of the SP of the icosahedron*)

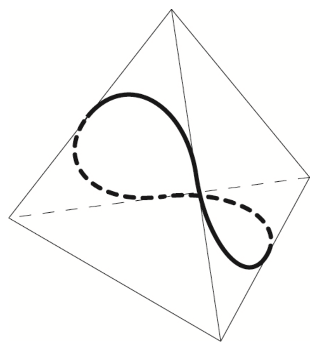


Fig. 43
$LC_{1/3}$ Pf the tetrahedron
(*CC of the SP of the tetrahedron*)

Based on the same criteria, here are the remaining eight platonic *LC* (figures 44-51):

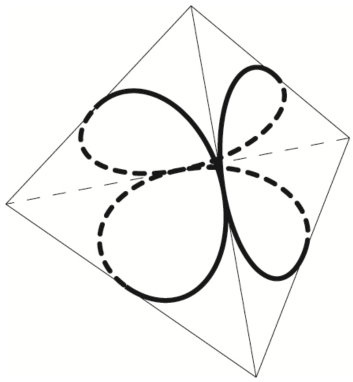


Fig. 44
$LC_{2/3}$ *of the tetrahedron*
(*CC of the SP of the tetrahedron*)

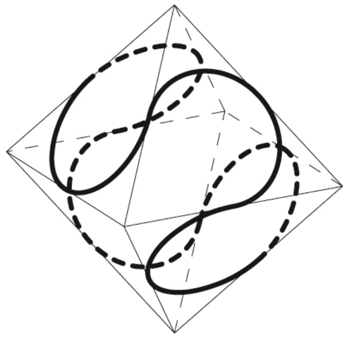


Fig. 45
$LC_{2/3}$ *of the octahedron*
(*CC of the SP of the hexahedron*)

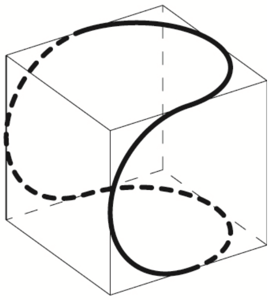


Fig. 46
$LC_{1/2}$ *of the hexahedron*
(*CC of the SP of the octahedron*)

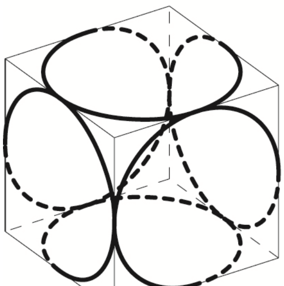


Fig. 47
$LC_{3/4}$ *of the hexahedron*
(***CC** of the SP of the octahedron*)

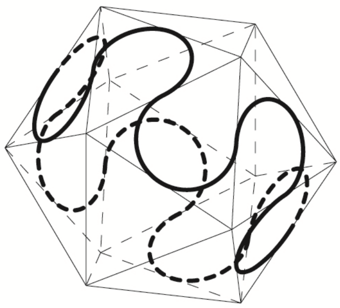


Fig. 48
$LC_{2/3}$ *of the icosahedron*
(*CC of the SP of the dodecahedron*)

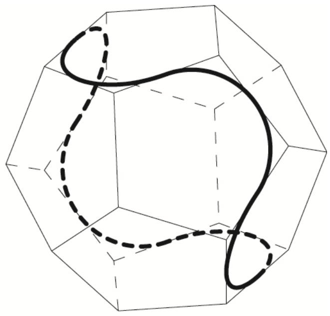


Fig. 49
$LC_{2/5}$ *of the* dodecahedron
(*CC of the SP of the icosahedron*)

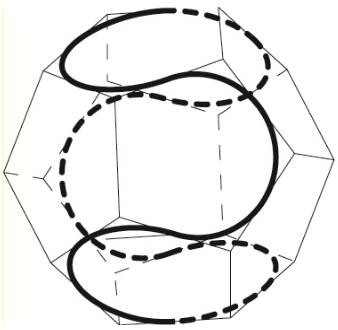


Fig. 50
$LC_{3/5}$ *of the* dodecahedron
(*CC of the SP of the icosahedron*)

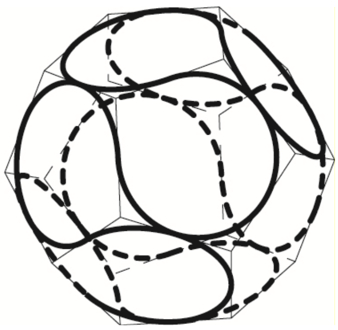


Fig. 51
$LC_{4/5}$ *of the* dodecahedron
(*CC of the SP of the icosahedron*)

Our study on the sphere can now be considered concluded, even if the temptation to want to say something else is always strong. For example, it would be interesting to proceed by describing the properties and the degree of symmetry of these last Platonic *LC*. It will be done on another occasion and, perhaps, even more, considering that there is still a lot to explore in the contest of the Spherical Paths. This is obviously my personal conviction, but imagine, just to suggest an idea, that you want to deal with them from a mechanical point of view. You never know where it ends!

Appendix: answers to the questions asked at the end of section 1.

- The probability space, where fixed n we take the pair (L, ϑ), is represented by the square $(0;\pi)\times(0;\pi]$, while the values that lead to the formation of a SP of frequency n are the pairs(L , ϑ) with $\vartheta \in (0;\pi]$ and $L = L(\vartheta)$, that is the couples of the function graphic $L = 2arcsin\left(\frac{\sin\frac{\pi}{n}}{\sin\frac{\theta}{2}}\right)$. The probability of being in any point of said curve is null, as this is given, as known, by the ratio between the null area of the curve and the area of the square $(0;\pi)\times(0;\pi]$ pari a π^2; on the other hand, the total probability will be the sum of the infinite null areas which, being of the cardinality of the countable (one for every fixed n), will also lead to a total probability zero.

- The probability of tracing a wave with a real compass, having points of a certain thickness and therefore confusing even if not exactly superimposed, will instead be not zero, since this time the values will be attributable not to a graph but to a strip of small area but not nothing; the probability in this case will depend on the ratio of the area of this to the surface area of the sphere. Therefore, it would seem that the SP has been traced in an apparently exact way (let's not forget the distinction between the real and the ideal).

- If, after the disappointing result illustrated in Figure 6, a second series of identical operations were continued, another Loop would be obtained out of phase with respect to the first (Figure 52); it should be noted that on the sphere points A_6 and B_6 will not coincide with A_1 and B_1 respectively, but with the backward points A_7 and B_7.

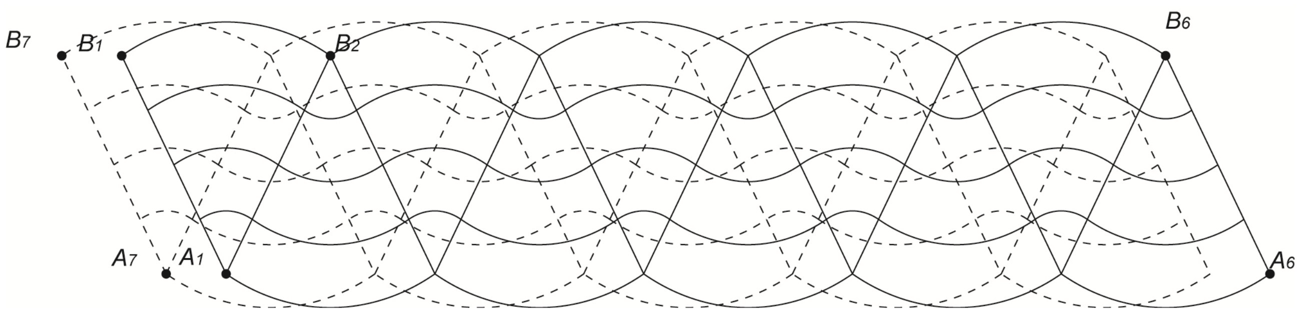


Fig. 52

11. **Cyclic Spherical Paths**.

So far we have considered the Spherical Paths which close after only one revolution along a maximum circumference and, therefore, with an even number of arcs. We would now like to draw inspiration from this last result to consider those Spherical Paths that close after a whole number of revolutions and which we will call *k*-cyclic. Under what conditions will these Spherical Paths close? By specifying now with *n* the number of arcs in a single turn (meaning by *n* a fraction of arcs of the SP), we will be able to observe that after *k* turns *nk* arcs will be described and that the SP will close when *nk* is even (that is when $nk = 2s$ with $s \in N$ and $s \geq 3$).

From this it follows that *n* will be equal to the ratio between two integers (in fact, $n = \frac{2s}{k}$, where *s* and *k* are the natural minima such that *n* can be expressed in the above way) and, therefore, *n* will be odd or rational. Let's make a first example of a SP with $n = 5$: this will close on condition that *nk* is even and, therefore, with $k = 2$ (it should be noted that this is valid for all odd numbers *n* and not only for 5); we will therefore have a SP that closes with $nk = 10$ arcs, ie after 10 rotations of the compass, corresponding to two turns along a maximum circle; after the first 5 rotations it will be out of phase by one step with respect to the starting configuration (figure 53) and it will be so up to the tenth rotation (figure 54), after which it will close.

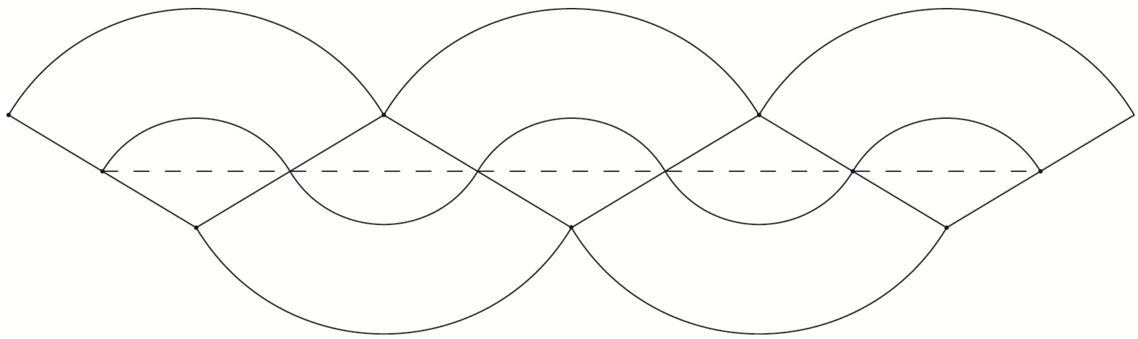


Fig. 53

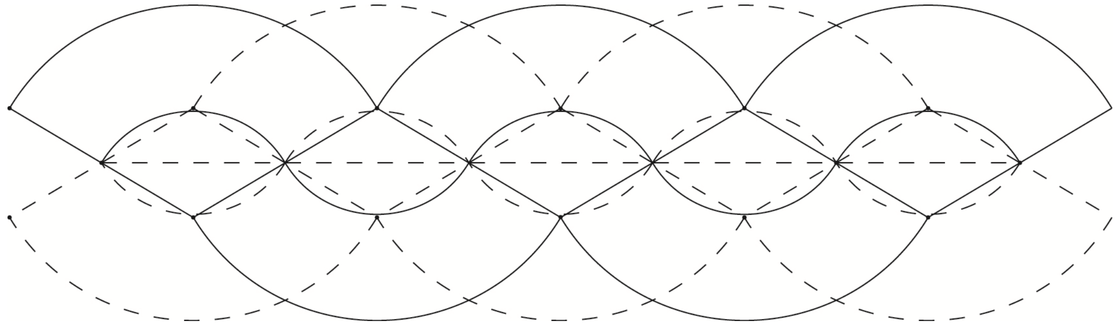


Fig. 54

By fixing instead a rational n, one can obtain k-cyclic Spherical Paths with $k \geq 3$. For example, setting $n = \frac{9}{2}$, we will have $\frac{9}{2}k$, with $k = 4$, from which $nk = 18$, and therefore have 4-cyclic SP (that is, they will close with 18 arcs, equivalent to 4 turns along a maximum circle). These will behave in the following way: after the first series of rotations, and until the end of the second series, they will be out of phase by ½ step, out of phase by one step during the third series, by ½ step during the fourth and, at the end of this last (i.e. the eighteenth rotation) will close.

Finally, if we fix an irrational n, for example $n = 2\pi$ (in this particular case, since the sphere has a unitary radius, it will be $p = 1$), we will obtain Spherical Paths which, as mentioned above, never close.

In general, the polyhedra connected to the complex Spherical Paths are antiprisms (these have already been mentioned in footnote 14). There are only four Spherical Paths in which the connected antiprism is a Platonic solid: the two of the tetrahedron and the two of the octahedron with $n = 6$ listed in Table 3; in fact, these are the only Spherical Paths in which the number n of the vertices corresponds to the number of the vertices of the Platonic solid from which they take their name. Of the other eight Platonic paths, two are connected to Archimedean antiprisms and, precisely, the first and fourth of the icosahedron, while the remaining six are connected to non-Archimedean antiprisms. However, it is observed that the antiprisms connected to these last eight Spherical Paths will have a precise relationship with the Platonic solid from which they take their name. For example, consider the Spherical Path of the hexahedron, in which the number n of vertices is $n = 6$; the antiprism connected to it is clearly shown in the center of the three figures below. This antiprism, which has two equilateral triangles as its base surface and six isosceles triangles as its side surface, is nothing more than a hexahedron with two truncations; these, as can be seen in the figures below, are obtained with two cross section planes parallel to each other, opposite with respect to the center of gravity of the solid and each passing through $\frac{n}{2} = \frac{6}{2} = 3$ vertices. In summary, the antiprism connected to a Platonic SP of n vertices can be one of the following solids: tetrahedron, octahedron, Archimedean or non-Archimedean antiprism; in the latter case it will be the equivalent of a Platonic solid (Hexahedron, dodecahedron or icosahedron) with two truncations obtained (in a way as described in figures 55-57) with two section planes each passing through $\frac{n}{2}$ vertices.

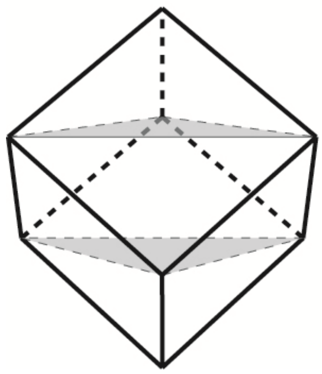


Fig. 55

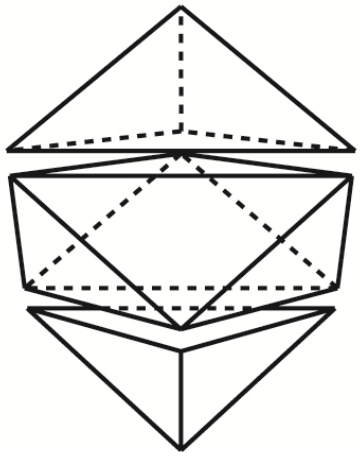


Fig. 56

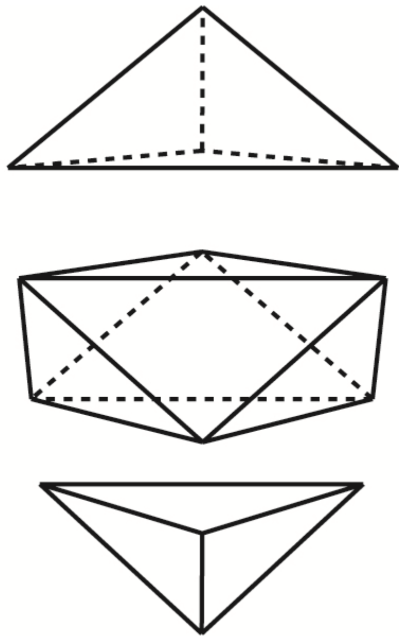


Fig. 57

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